ON A SOLUTION OF THE GENERAL BIQUADRATIC EQUATION.

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Very often in mathematical work does one wish to write out without waste of time the value of the unknown in a given biquadratic equation. Nowhere in text-books or mathematical writings do I find the solution to a biquadratic given in such form that one by merely substituting in a formula may get the roots. I have found the formula here given convenient and I do not know that the formula or this particular method of getting the result has ever before been published.

Let the general biquadratic be

$$x^4+a_1x^3+a_2x^2+a_3x+a_4=0,$$

and let the roots be a, b, c, d. Then follow, as is well known,

$$\begin{aligned} a+b+c+d&=-a_1,\\ ab+ac+ad+bc+bd+cd&=a_2,\\ abc+abd+acd+bcd&=-a_3,\\ abcd&=a_4. \end{aligned}$$

Now let

$$\left.\begin{aligned} z_1&=ab+cd\\ z_2&=ac+bd\\ z_3&=ad+bc \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{I.}$$

Then it follows that,

$$\begin{aligned} z_1+z_2+z_3&=a_2,\\ z_1z_2+z_1z_3+z_2z_3&=(ab+cd)(ac+bd)+(\quad)(\quad)+(\quad)(\quad)\\ &=a^2bc+ab^2d+c^2ad+cbd^2+\ldots\ldots+\ldots\ldots\\ &=\Sigma a^2bc=a_1a_3-4a_4, \end{aligned}$$

and

$$\begin{aligned} z_1z_2z_3&=(ab+cd)(ac+bd)(ad+bc)\\ &=\Sigma a^3bcd+\Sigma a^2b^2c^2\\ &=a_3^2+a_1^2a_4-4a_2a_4. \end{aligned}$$

Then z_1, z_2, z_3 are therefore the roots of the reducing cubic :

$$z^3-a_2z^2+(a_1a_3-4a_4)z-(a_3^2+a_1^2a_4-4a_2a_4)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{II.}$$

Now from I we have

$$z_1^2 = a^2b^2 + c^2d^2 + 2abcd$$
$$= a^2b^2 + c^2d^2 + 2a_4,$$
$$\therefore\ z_1^2 - 4a_4 = a^2b^2 + c^2d^2 - 2abcd = (ab-cd)^2,$$
$$\therefore\ \sqrt{z_1^2 - 4a_4} = ab - cd \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (a),$$
$$\text{but } z_1 = ab + cd \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (b).$$

Therefore by adding (a) and (b),

$$ab = \tfrac{1}{2}\{z_1 + \sqrt{z_1^2 - 4a_4}\} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (c),$$

and by subtracting (a) from (b) we have

$$cd = \tfrac{1}{2}(z_1 - \sqrt{z_1^2 - 4a_4}) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (d).$$

In the same manner we get

$$ac = \tfrac{1}{2}(z_2 + \sqrt{z_2^2 - 4a_4}) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (e),$$
$$bd = \tfrac{1}{2}(z_2 - \sqrt{z_2^2 - 4a_4}) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (f),$$
$$ad = \tfrac{1}{2}(z_3 + \sqrt{z_3^2 - 4a_4}) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (g),$$
$$bc = \tfrac{1}{2}(z_3 - \sqrt{z_3^2 - 4a_4}) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (h).$$

But $ab + ac + ad = a(b + c + d)$

$$= (-a_1 - a)a, \text{ since } b + c + d = -a_1 - a$$
$$= -a^2 - a_1 a.$$

Also $ab + ac + ad = \frac{1}{2}\{z_1 + z_2 + z_3 + \sqrt{z_1^2 - 4a_4} + \sqrt{z_2^2 - 4a_4} + \sqrt{z_3^2 - 4a_4}$ from (c), (e), and (g). Therefore,

$$a^2 + a_1 a + \tfrac{1}{2}\{a_2 + \sqrt{z_1^2 - 4a_4} + \sqrt{z_2^2 - 4a_4} + \sqrt{z_3^2 - 4a_4}\} = 0,$$

which is a biquadratic equation giving the value of one root a, *i. e.*

$$a = \frac{-a_1 \pm \sqrt{a_2 - 2\{a_2 + \sqrt{z_1^2 - 4a_4} + \sqrt{z_2^2 - 4a_4} + \sqrt{z_3^2 - 4a_4}\}}}{2} \ \ldots\ldots$$

The four roots are, therefore,

$$\left.\begin{matrix} a \\ b \\ c \\ d \end{matrix}\right\} = \tfrac{1}{2}\left\{ -a_1 \pm \sqrt{a_1^2 - 2\{a_2 \pm \sqrt{z_1^2 - 4a_4} \pm \sqrt{z_2^2 - 4a_4} \pm \sqrt{z_3^2 - 4a_4}} \right. \ \ldots\ldots \text{III},$$

where the sequence of signs under the main radical, as can be seen from formulæ (c) to (h), is

$$\begin{array}{lccc}
\text{for } a, & + & + & + \\
\text{for } b, & + & - & - \\
\text{for } c, & - & + & - \\
\text{for } d, & - & - & +
\end{array}$$

For the z_1, z_2, z_3 in this solution III must be substituted the roots of the cubic II.

Example. As an example take the biquadratic

$$x^4-x^3-7x^2+x+6=0.$$

Here we have,

$$a_1=-1, \quad a_3=1,$$
$$a_2=-7, \quad a_4=6,$$

from which the cubic becomes $z^3+7z^2-25z-175=0$, of which the roots are 5, -7, and -5. Thus the roots of the biquadratic are

$$\tfrac{1}{2}\{1\pm\sqrt{1-2\{-7\pm1\pm5\pm1\}},$$

or 1, -1, -2, 3, which are seen to be correct.

Care must be exercised that the proper sign before the main radical is taken.

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EQUATION OF PAYMENTS.

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Let it be required to find the equated time of two payments, P and P_1, due at the end of t and t_1 years respectively, and r being the rate of interest.

Represent the equated time by x when $t>t_1$.

I. By Simple Interest.

1st Method. The discount on P for $(t-x)$ years must equal the interest on P_1 for $(x-t_1)$ years.

$\dfrac{P(t-x)r}{1+(t-x)r}$=discount on P due $(t-x)$ years hence.

$P_1(x-t_1)r$=interest on P_1 for $(x-t_1)$ years.

$$\therefore \frac{P(t-x)r}{1+(t-x)r}=P_1(x-t_1)r.$$